NEGATIVE UNADJUSTED SUM OF SQUARES IN THE ANALYSIS OF VARIANCE

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The problem of a negative unadjusted sum of squares in the analysis of variance is no mere academic question to the social scientist when one suddenly appears in the analysis of his pet project. Often the experience is, at best, confusing and frustrating, leading to hours of recalculation and literature research. This paper was developed as an attempt to clarify the problem for the experimentally-minded social scientist and as an admonition to those who might be tempted to develop experiments too recklessly. A simple proof of negative unadjusted sums of squares is presented in a format typical of the statistical references commonly used by many social scientists. The authors believe the proof to be original although certainly the subject has been more elegantly treated by Anscombe (1), Nelder (2), or Thompson (3). These articles, however, are found in the Journal of the Royal Statistical Society, Biometrika, and the Annals of Mathematical Statistics. Likely not even the most optimistic among statisticians would claim that these journals are widely read, used as resource references, or even understood by most social science researchers.

The authors became aware of the problem when a graduate student was completing his thesis in social psychology. He was utilizing a four-factor, completely crossed factorial design which we will use as an example in the proof. While running his analysis, a negative sum of squares occurred in a triple interaction term. His thesis committee was skeptical (as thesis committees often tend to be) and he was summarily returned to the calculator; the negative sum of squares, however, persisted. The following proof was developed as a result of the unwillingness of that negative estimate to disappear.

The authors feel the notational system used to be consistent with those often encountered and used by social scientists [e.g., Winer (4), Myers (5)]. The terms necessary to derive an ABC interaction with at least three main factors are:

(1) $$T = \frac{\left(\sum_{i}^{n}\sum_{j}^{a}\sum_{k}^{b}\sum_{o}^{c} Y\right)^2}{n\,a\,b\,c},$$

(2) $$A = \frac{\sum_{j}^{a}\left(\sum_{i}^{n}\sum_{k}^{b}\sum_{o}^{c} Y\right)^2}{n\,b\,c},$$

(3) $$B = \frac{\sum_{k}^{b}\left(\sum_{i}^{n}\sum_{j}^{a}\sum_{o}^{c} Y\right)^2}{n\,a\,c},$$

(4) $$C = \frac{\sum_{o}^{c}\left(\sum_{i}^{n}\sum_{j}^{a}\sum_{k}^{b} Y\right)^2}{n\,a\,b},$$

(5) $$AB = \frac{\sum_{j}^{a}\sum_{k}^{b}\left(\sum_{i}^{n}\sum_{o}^{c} Y\right)^2}{n\,c},$$

(6) $$AC = \frac{\sum_{j}^{a}\sum_{o}^{c}\left(\sum_{i}^{n}\sum_{k}^{b} Y\right)^2}{n\,b},$$

(7) $$BC = \frac{\sum_{k}^{b}\sum_{o}^{c}\left(\sum_{i}^{n}\sum_{j}^{a} Y\right)^2}{n\,a},$$

(8) $$ABC = \frac{\sum_{j}^{a}\sum_{k}^{b}\sum_{o}^{c}\left(\sum_{i}^{n} Y\right)^2}{n},$$

and

(9) k, p = constants

The sums of squares for terms (2) through (8) are found in the following manner:

(10) $SS_A = A-T$,

(11) $SS_B = B-T$,

(12) $SS_C = C-T$,

(13) $SS_{AB} = AB-A-B+T$,

(14) $SS_{AC} = AC-A-C+T$,

(15) $SS_{BC} = BC-B-C+T$,

and

(16) $SS_{ABC} = ABC-AB-AC-BC+A+B+C-T$.

The inequality necessary to generate a negative sum of squares for the ABC interaction is

(17) $ABC+A+B+C < AB+AC+BC+T$

Conditions observed in the student's thesis and required for deriving the above inequality are:

(18) terms (1) through (9) $\geq$ o,

(19) a minimum of three main factors,

(20) unequal cell n's,

(21) $A \leq T+k$,

(22) $C+(k+p) < BC$

(23) $ABC+B < AB+AC+p$

Adding inequalities yields:

(24) $ABC+B+A+C+(k+p) < AB+AC+BC+T+(k+p)$

Adding a negative (k+p) to each side of the inequality produces the original inequality (17):

(17) $ABC+A+B+C < AB+AC+BC+T.$

In the student's data the double interactions achieved a magnitude outweighing the contribution of the other components. Unequal cell sizes seem of particular importance in producing the negative sum of squares because of the concomitant increase in the probability of heterogeneity of variance, a violation of a basic, and too often casually treated, assumption in the analysis of variance. The occurrence of the negative estimate is an indication that the design was inappropriate and that the data should have been analyzed differently, or better, that the experiment should have been more carefully planned and conducted.

The authors hope this paper will save future researchers hours of fruitless searching for nonexistent errors in calculation. More importantly they hope it will encourage more care in the design and analysis of experiments. While such techniques as the Doolittle method of least squares analysis, which minimize the interaction terms, tend to eliminate the possibility of negative sums of squares, there is likely no substitute for more rigorous attention to the assumptions underlying the analysis.

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